The appropriate boundary conditions on Ψ in the various regions are a) regions 2 and 6

$$\left.\frac{\partial\Psi_{2,6}}{\partial y_1}\right|_{y_1=\epsilon} = \mu\frac{N_e}{l_e}\begin{bmatrix}I_2\\I_6\end{bmatrix}e^{j\omega t}$$
$$\left.\frac{\partial\Psi_{2,6}}{\partial y_1}\right|_{y_1=-\epsilon} = 0 \tag{13}$$

where N_e is the number of turns of wire in an end coil of length l_e carrying a current $I_{2,6}$ (amps), and b) regions 1, 3, 5, and 7

$$\left.\frac{\partial\Psi_{1,3,5,7}}{\partial y_1}\right|_{y_1=\pm\epsilon} = 0 \tag{14}$$

Solving Eq. (10) subject to the aforementioned boundary conditions gives a) in regions 2 and 6

$$\Psi_{2,6} = \frac{\mu N_e}{2\epsilon l_e}\begin{bmatrix}I_2\\I_6\end{bmatrix}e^{j\omega t}\left[\epsilon^2\left(\frac{y^2}{2}+y\right) - \left(\frac{x_1^2}{2} + a_1{}^{(\pm)}x_1 + a_2{}^{(\pm)}\right)\right] \tag{15}$$

b) in regions 3 and 5,

$$\Psi_{3,5} = -(b_1{}^{(\pm)}x_1 + b_2{}^{(\pm)})e^{j\omega t} \tag{16}$$

and c) in regions 1 and 7

$$\Psi_{1,7} = -c_1{}^{(\pm)}e^{j\omega t} \tag{17}$$

These solutions may be varified by a direct application of the integral form of Ampere's Law assuming, as is consistent with Eqs. (8) and (9), that B_y is independent of y. The general solution in regions 1 and 7 is identical to that in regions 3–5; however, since the former are semi-infinite, and all fields must be bounded, Ψ cannot be linear in x_1 there and hence, reduces to a simple constant.

Also, consistent with the accuracy of the solutions in region 4 (zeroth-order in ϵ) the y dependence of Ψ in regions 2 and 6 (second-order in ϵ) may be neglected. Then the cross-channel magnetic field and the azimuthal electric field may be determined from Eqs. (11) and (12).

Matching these fields in x_1 at the several region boundaries leads to twelve algebraic equations in the twelve unknowns $a_1{}^{(\pm)}$, $a_2{}^{(\pm)}$, $b_1{}^{(\pm)}$, $c_1{}^{(\pm)}$ and I_2 and I_6. Choosing I_2 and I_6 arbitrarily leads to an overspecified system of equations. The two missing constants, needed to close this system of equations, describe additional fields inside the generator and correspond to the major components of the undesirable end solutions.[1,2] It is only when I_2 and I_6 take on precisely the values determined by the previously mentioned matching procedure that these additional fields are mathematically unnecessary and are, in fact, absent in the generator.

The magnitudes of the desired end currents determined by this procedure are

$$I_2 = 2\epsilon B_0/k\mu N_e \tag{18}$$

and

$$I_6 = -2\epsilon B_0 e^{-jkL}/k\mu N_e \tag{19}$$

If the generator is so designed as to have an integral number of wavelengths λ of the magnetic field in the length L then

$$k = 2\pi/\lambda \text{ and } e^{-jkL} = 1 \tag{20}$$

For this case

$$\begin{bmatrix}I_2\\I_6\end{bmatrix} = \pm\frac{2\epsilon B_0}{k\mu N_e} = \pm\frac{2r_0 B_0}{\mu N_e} \tag{21}$$

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Analysis of Short Beams

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Nomenclature

A	= area of cross section of the beam
b,d	= breadth and depth of the rectangular beam
b_2,B_{22},C_{22}	= cross-sectional constants
D	= slenderness ratio = L/d
D^*	= L/nd
E	= Young's modulus
G	= shear modulus
I	= moment of inertia
k	= $(E/G)^{1/2}$
L	= length of the beam
n	= number of half waves into which the column buckles
P	= axial load
P_E	= Euler load
q	= transverse loading on the beam
U	= strain energy
v,w	= displacements along x and y directions, respectively
v_b	= transverse deflection caused by bending
v_s	= $v - v_b$
V	= nondimensional tip deflection = $3vEI/P_tL^3$
W	= work
w_n	= nth warp function
y,z	= Car esian coordinates
β	= shear coefficient
σ_z,σ_{zy}	= direct and shear stresses
ϕ_n	= axial variation of nth warp function
λ	= P/P_E
$'$	= denotes differentiation with respect to z

Introduction

The beam may be defined as a slender structural component with one of its dimensions much larger than the other two. Its structural behaviour may be satisfactorily approximated by the elementary theory of bending as long as the slenderness ratio is sufficiently large. But for short beams, Euler's theory is inadequate because of the substantial influence of the secondary effects. In 1921, Timoshenko[1] extended the domain of validity of the beam theory for vibrations by incorporating the effect of transverse shear into the differential equation. Later this theory was extended for the stability[2] and static[3] analysis of beams. Recently the author has proposed a new formulation for the vibration analysis of short beams.[4] In this Note, we extend this formulation for the static and stability analysis of short beams and study some typical examples. A feature noted in Ref. 4, namely the use of refined shear coefficient in Timoshenko theory leading to increased discrepancies, is also noticed in the static and stability analysis of beams.

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Stability of Columns

The basic assumption is that the direct strains in the plane of the cross section are zero. The effect of Poisson's ratio is ignored. The stresses, strains, and displacements are assumed not to vary along the axis normal to the plane of bending. Taking x and y axes as principal axes and considering the flexural instability in the plane of YOZ, we have, by virtue of the aforementioned assumptions, all other stresses zero except σ_z and σ_{zy}. The strain energy caused by bending is

$$U = \frac{1}{2}\int_0^L \int_{-d/2}^{d/2} (\sigma_z\epsilon_z + \sigma_{zy}\epsilon_{zy})bdydz \tag{1}$$

and the work done by the compressive loads P, during the application of the disturbance v is

$$W = \frac{1}{2}\int_0^L P\left(\frac{dv}{dz}\right)^2 dz \tag{2}$$

Using the stress-strain relations

$$\sigma_z = E\partial w/\partial z,\ \sigma_{zy} = G(\partial w/\partial y + dv/dz) \tag{3}$$

and the principle of stationary value of total potential energy, the equations governing the stability of columns may be obtained as

$$(GA - P)\frac{d^2v}{dz^2} = -G\frac{d}{dz}\int_{-d/2}^{d/2}\frac{\partial w}{\partial y}bdy$$

and

$$E\partial^2 w/\partial z^2 + G\partial^2 w/\partial y^2 = 0 \tag{4}$$

Boundary conditions at each end can be either

$$v = 0$$

or

$$(GA - P)\frac{dv}{dz} + G\int_{-d/2}^{d/2}\frac{\partial w}{\partial y}bdy$$

either

$$w = 0$$

or

$$E\partial w/\partial z = 0$$

and

$$G(\partial w/\partial y + dv/dz) = 0 \text{ at free edges} \tag{5}$$

When w is symmetric about y axis, there is no coupling between w and v and Eqs. (4) reduce to

$$(GA - P)d^2v/dz^2 = 0 \tag{6}$$

with boundary conditions at ends as

$$\text{either } v = 0 \text{ or } v' = 0 \tag{7}$$

Eq. (6) yields a solution that a cantilever column subjected to an axial load $P = GA$ is unstable and the failure is by shearing of cross sections. Nevertheless, in many cases, bending mode of instability precedes this mode and so this solution is of little practical importance.

When w is symmetric, we can express w as[4]

$$w(y,z) = -y\frac{dv_b}{dz} - \sum_{n=2}^{\infty}\bar{w}_n(y)\phi_n(z) \tag{8}$$

where

$$\bar{w}_n = -\iint \bar{w}_{n-1}dydy$$

and (9)

$$\bar{w}_2 = \iint ydydy$$

The constants of integration in Eqs. (9) are evaluated from the zero free-edge-strain conditions. Using Eq. (8), the expression for strain energy becomes

$$U = \frac{1}{2}\int_0^L\int_{-d/2}^{d/2} E\left[-v_b'' - \sum_{n=2}^{N}\bar{w}_n(y)\phi_n'(z)\right]^2 + \frac{1}{2}\int_0^L\int_{-d/2}^{d/2} G\left[-\sum_{n=2}^{N}\frac{d\bar{w}_n(y)}{dy}\phi_n(z)\right]^2 + \frac{1}{2}\int_0^L GA\,(v_s')^2dz \tag{10}$$

The last term in the above expression is to account for the strain energy associated with shearing of cross sections. The work done by the compressive loads becomes

$$W = \frac{1}{2}\int_0^L P\left[\frac{d}{dz}(v_b + v_s)\right]^2 dz \tag{11}$$

Retaining various number of terms in the expansion for w and using the principle of stationary value of total potential energy governing equations to various orders of approximation can be deduced. It can be easily verified that the equations obtained retaining first-two-terms in Eq. (8) correspond to the Timoshenko equation, with the value of shear coefficient equal to unity. Retaining first-three-terms in Eq. (8), we get the governing equations as

$$\begin{aligned}(GA\text{-}P)v_s'' - v_b'' &= 0\\ E(Iv_b^{iv} + b_2\phi_2''') + P(v_s'' + v_s'') &= 0\\ E(b_2v_b''' + B_{22}\phi_2'') + GC_{22}\phi_2 &= 0\end{aligned} \tag{12}$$

and these will be referred to hereafter as second-order approximation equations.

Beams under Static Transverse Loads

The strain energy expression, for a beam subjected to static loading of q lbs/in., is the same in Eq. (1). But the expression for work done becomes

$$W = \int_0^L q(z)vdz + (qv)_{z=0} + (qv)_{z=L} \tag{13}$$

From the principle of stationary value of total potential energy, the governing equations may be obtained as

$$\begin{aligned}GAv'' + q &= -G\frac{d}{dz}\int_{-d/2}^{d/2}\frac{\partial w}{\partial y}bdy\\ E(\partial^2 w/\partial z^2) + G\partial^2 w/\partial y^2 &= 0\end{aligned} \tag{14}$$

Boundary conditions at the end can be

either $v = 0$

or $G\left(Av' + \int_{-d/2}^{d/2}\frac{\partial w}{\partial y}bdy\right) = -q_e$

either $w = 0$

or $E\partial w/\partial z = 0$

Table 1 Comparison of the critical load parameter λ

D^*	By Timoshenko theory with the shear coefficient equal to 0.666	0.870	Present work (second-order approximation)
2	0.550261	0.614892	0.435977
4	0.821221	0.864622	0.769604
6	0.916747	0.934938	0.882511
8	0.951400	0.962331	0.933448
10	0.968342	0.975562	0.954253
20	0.991893	0.993776	0.988152
40	0.997960	0.998437	0.997009
100	0.999673	0.999750	0.999520

Table 2 Comparison of the tip deflection V

D	By Timoshenko theory with the shear coefficient equal to 0.666	0.870	Present work (second-order approximation)
2	1.24844	1.19037	1.34094
4	1.06211	1.04759	1.08751
6	1.02760	1.02115	1.03927
8	1.01553	1.01190	1.02205
10	1.00994	1.00761	1.01413
20	1.00248	1.00190	1.00354
40	1.00062	1.00046	1.00089
100	1.00010	1.00008	1.00014

and

$$G(\partial w/\partial y + dv/dz) = 0 \text{ at free edges} \tag{15}$$

Following the procedure of previous section, Timoshenko equation can be shown to be a special case of this formulation and the second-order approximation equation can be obtained as

$$\begin{aligned} GAv_s'' + q &= 0 \\ E(Iv_b^{iv} + b_2\phi_2''') &= q \\ E(b_2v_b''' + B_{22}\phi_2'') + GC_{22}\phi_2 &= 0 \end{aligned} \tag{16}$$

Results and Discussion

Two simple examples are considered in order to bring out the necessity of the present formulation in the short column range, 1) stability of a simply-supported rectangular column, and 2) cantilever rectangular beam subjected to an end load. The cross-sectional constants involved in second-order approximation equations can be evaluated as[4]

$$\begin{aligned} A &= bd \qquad I = (\tfrac{1}{12})bd^3 \\ b_2 &= -C_{22} = (\tfrac{1}{120})bd^5 \qquad B_{22} = (\tfrac{17}{20160})bd^7 \end{aligned} \tag{17}$$

By using second-order approximation equations one can obtain the critical load parameter of a simply-supported rectangular column as

$$\lambda = 1/[1 + k^2\pi^2/12D^{*2} + 84k^2\pi^2/(k^2\pi^2 + 10D^{*2})] \tag{18}$$

whereas Timoshenko theory yields

$$\lambda = 1/[1 + (k^2\pi^2/12\beta D^{*2})] \tag{19}$$

and the value of λ is unity by Euler's theory. Comparison of results obtained by various theories is shown in Table 1.

In the case of a cantilever rectangular beam subjected to a load P_t at the free end, the nondimensional deflection at the free end V by second-order approximation equations is

$$V = \left[\frac{\cosh\mu}{\cosh\mu + 84}\right]\left[1 + \frac{252}{\mu^2}\left(1 - \frac{\tanh\mu}{\mu}\right)\right]$$

with $\mu = [(840G/E)D^2]^{1/2}$

Timoshenko theory gives

$$V = 1 + (E/4\beta GD^2)$$

and the value is unity by elementary theory.

Comparison of results obtained by various theories is given in Table 2. In both the cases the results indicate that use of refined shear coefficient in Timoshenko equation increases the difference in the results from the more accurate results by the second-order approximation equations.

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An Efficient Triangular Shell Element

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VARIOUS investigators have analyzed arbitrary shells by using curved elements. Gallagher[1] has presented an exhaustive review of available plate and shell elements. Though the development of membrane elements is almost complete, the existing bending elements may still be replaced by more efficient formulations. Oden and Wempner[2] suggested to use the linear shear theory and to neglect the shear energy. A discrete equivalent of the Kirchhoff assumptions is then introduced over the element in order to relate rotations and the transverse displacement. Recently, a family of triangular shell elements, based on the discrete Kirchhoff assumptions, has been presented by the present author.[3]

In this study, a new triangular shell element is developed by using the linear shear theory. The shear energy is neglected and a discrete version of the Kirchhoff assumptions is introduced over the element.

1. Formulation of the Element

The true geometry $z(x,y)$ of the triangular element has been approximated by a shallow quadratic surface as described by Bonnes et al.[4] The strain-displacement relations for a thin shallow element are defined as follows:

$$\epsilon = e_m + \zeta\kappa \tag{1a}$$

$$\begin{bmatrix} e_{m1} \\ e_{m2} \\ e_{m12} \end{bmatrix} = \begin{bmatrix} u_{,x} - z_{,xx}w \\ v_{,y} - z_{,yy}w \\ u_{,y} + v_{,x} - 2z_{,xy}w \end{bmatrix} = e_m \tag{1b}$$

$$\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix} = \begin{bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{bmatrix} = \kappa \tag{1c}$$

where u, v, and w are displacements along ζ_1, ζ_2, and ζ_3 directions of a right-hand orthogonal coordinate system. β_x and β_y are rotations of the normal ζ_3 along ζ_1 and ζ_2 directions, respectively, ζ is the thickness coordinate of the element such that $-h/2 < \zeta < h/2$, and h is the thickness of the element.

By definition, the membrane forces and the bending moments are

$$\begin{aligned} (N_xN_yN_{xy}) &= he_m{}^TD \\ (M_xM_yM_{xy}) &= (h^3/12)\kappa^TD \end{aligned} \tag{2}$$

where the superscript T indicates the transpose of a vector or a matrix

$$D = E/(1-\nu^2)\begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & (1-\nu)/2 \end{bmatrix}$$

E is equal to the modulus of elasticity of the material, and ν is equal to Poisson's ratio of the material.

The internal energy of the element is defined by

$$U = \tfrac{1}{2}\int e_m{}^TDe_m dv + \tfrac{1}{2}\int \kappa^TD\kappa\zeta^2 dv \tag{3}$$

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